plotted together with the present one (dashed line) for $m = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. As regard to the steady state value, the numerical factor $2(\sqrt{3})/3$ appears instead of

$$\Gamma\left(\frac{1}{1+m}\right)\Bigg/\left[(m+1)^{\frac{1}{2}}\,\Gamma\left(\frac{m+3}{2m+2}\right)\right].$$

CONCLUSIONS

The integral method has been applied to the study of transient heat transfer to boundary layer flows with zero Prandtl number. Analytical solutions have been obtained in a very simple way for arbitrary potential flows and have been compared with Soliman–Chambrè [1] exact solution in the particular case of wedge flows. The overall accuracy of the obtained solution seems satisfactory and gives further confidence in the possibility of application of the integral method to this kind of time-dependent problem. (A similar conclusion was drawn by Stewartson [4] in the problem of the impulsive motion of a flat plate in a viscous fluid.)

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ON THE PERTURBATION SOLUTION OF THE ICE–WATER LAYER PROBLEM

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NOMENCLATURE

θ,	departure from freezing temperature;
ϕ,	normalized temperature (θ/θ_c);
X,	distance from fixed cooled surface;
x,	normalized distance (X/X_c);
t,	time;
τ,	normalized time (t/t_c);
θ_c,	representative temperature difference;
X_c,	final layer thickness or "active zone" depth;
t_c,	characteristic time $(\rho L X_c^2/K\theta_c$ or periodic time);
β,	normalized ice thickness;
C_p,	specific heat at constant pressure;
L,	latent heat of fusion;
ρ,	ice density;
K,	thermal conductivity;
Ste,	Stefan number $(C_p\theta_c/L)$.

In a previous paper [1], a perturbation solution was developed for the formation of an ice layer at the edge of a semi-infinite domain of water, initially at the freezing point, and subject to a prescribed variation in surface temperature. In this form the problem was posed as the solution of the equations

$$\frac{\partial^2\phi}{\partial x^2} = Ste\frac{\partial\phi}{\partial\tau} \tag{1}$$

and

$$\frac{d\beta}{d\tau} = \left(\frac{\partial\phi}{\partial x}\right)_\beta \tag{2}$$

subject to suitable initial and boundary conditions: one of the boundary conditions has since been found to be incorrect.

A regular perturbation expansion in *Ste* was then used for both the temperature $\phi(x, \tau, Ste)$ and interface location $\beta(\tau, Ste)$. For two particular surface temperature variations (sinusoidal and power law) the interface expressions took particular forms which, it has since been noticed, are each of the more general form

$$\beta(\tau, Ste) = \beta(\tau, 0)\left[1 + \frac{Ste}{6}\phi^\circ(\tau) + 0(Ste^2)\right]$$

where $\phi^\circ(\tau)$ is the particular surface temperature variation. This is clearly a very simple and convenient result and therefore it is worthwhile examining the problem to see whether the result applies to any other surface temperature variations and to ascertain the extent of the error incurred through the use of an incorrect boundary condition.

Taking

$$\phi(x, \tau, Ste) = \sum_{p=0}^{\infty} Ste^p\,\phi_p(x, \tau), \tag{3}$$

and

$$\beta(\tau, Ste) = \sum_{p=0}^{\infty} Ste^p \, \beta_p(\tau), \tag{4}$$

it follows that

$$\phi(\beta, \tau, Ste) = \sum_{p=0}^{\infty} Ste^p \, \phi_p(\beta, \tau).$$

But by virtue of equation (4)

$$\phi_p(\beta, \tau) = \phi_p(\beta_0, \tau) + \left(\frac{\partial \phi_p}{\partial x}\right)_{\beta_0} Ste\, \beta_1(\tau) + O(Ste^2)$$

and hence if $\phi(\beta, \tau, Ste) = 0$

$$\phi_0(\beta_0, \tau) = 0$$

$$\phi_1(\beta_0, \tau) + \beta_1(\tau) \frac{\partial \phi_0}{\partial x}(\beta_0, \tau) = 0, \quad \text{etc.} \tag{5}$$

The second of equations (5) differs from the earlier statement [1] which placed $\phi_1(\beta, \tau) = 0$. With this exception, the solution of equation (1) using equations (3) and (4) follows [1] exactly and leads to the temperature profiles

$$\phi_0(x, \tau) = \phi^\circ(\tau) \left[1 - \frac{x}{\beta_0}\right]$$

and

$$\phi_1(x, \tau) = \phi^\circ(\tau) \left[\left(\frac{\beta_1}{\beta_0^2} + \frac{\beta_0^2}{3!\phi^\circ} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\phi^\circ}{\beta_0}\right) - \frac{\beta_0}{2!\phi^\circ} \frac{\mathrm{d}\phi^\circ}{\mathrm{d}\tau}\right) x + \frac{1}{2!\phi^\circ} \frac{\mathrm{d}\phi^\circ}{\mathrm{d}\tau} x^2 - \frac{1}{3!\phi^\circ} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\phi^\circ}{\beta_0}\right) x^3\right].$$

In the solution of equation (2) it is important to recognize that

$$\left(\frac{\partial \phi_n}{\partial x}\right)_{\beta} = \left(\frac{\partial \phi_n}{\partial x}\right)_{\beta_0} + \left(\frac{\partial^2 \phi_n}{\partial x^2}\right)_{\beta_0} Ste\, \beta_1(\tau) + O(Ste^2).$$

Hence equation (2) may be re-stated as

$$\frac{\mathrm{d}\beta_0}{\mathrm{d}\tau} = \left(\frac{\partial \phi_0}{\partial x}\right)_{\beta_0} \tag{6}$$

$$\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} = \left(\frac{\partial \phi_1}{\partial x}\right)_{\beta_0} + \beta_1 \left(\frac{\partial^2 \phi_0}{\partial x^2}\right)_{\beta_0} \quad \text{etc.} \tag{7}$$

which are to be solved subject to $\beta_p(\tau) = 0$.

Equation (6) may be solved immediately giving

$$\beta_0(\tau) = \{-2 \int_0^{\tau} \phi^\circ(\tau) \, \mathrm{d}\tau\}^{\frac{1}{2}}. \tag{8}$$

Using the temperature profiles, equation (7) becomes

$$\frac{\mathrm{d}\beta_1}{\mathrm{d}\tau} = \beta_1 \frac{\phi^\circ}{\beta_0^2} + \frac{\beta_0}{6} \frac{\mathrm{d}\phi^\circ}{\mathrm{d}\tau} + \frac{\phi^\circ}{3} \frac{\mathrm{d}\beta_0}{\mathrm{d}\tau}$$

the first term on the right-hand side of which has been added by virtue of the modification to equation (5). The solution of this non-homogeneous linear equation is readily found by re-writing it [using equation (8)] in the form

$$\frac{\mathrm{d}\beta_1}{\mathrm{d}\beta_0} + \frac{\beta_1}{\beta_0} = \frac{\phi^\circ}{3} - \frac{\beta_0^2}{6\phi^\circ} \frac{\mathrm{d}\phi^\circ}{\mathrm{d}\tau}$$

which integrates to give

$$\beta_1(\tau) = \frac{1}{\beta_0} \int_0^{\beta_0} \beta_0 \left[\frac{\phi^\circ}{3} + \frac{\beta_0}{6} \frac{\mathrm{d}\phi^\circ}{\mathrm{d}\beta_0}\right] \mathrm{d}\beta_0.$$

But

$$\frac{1}{6} \frac{\mathrm{d}}{\mathrm{d}\beta_0} (\beta_0^2 \phi^\circ) = \frac{\beta_0^2}{6} \frac{\mathrm{d}\phi^\circ}{\mathrm{d}\beta_0} + \frac{\phi^\circ}{3} \beta_0$$

and hence

$$\beta_1(\tau) = \frac{\beta_0 \phi^\circ}{6}.$$

Therefore

$$\beta(\tau, Ste) = \beta_0(\tau) + Ste\, \beta_1(\tau) + O(Ste^2)$$
$$= \beta(\tau, 0) \left[1 + \frac{Ste}{6} \phi^\circ(\tau) + \ldots\right]$$

and thus showing that the result holds for arbitrary excursions of the surface temperature.

This simple result is especially useful for ice–water systems, where *Ste* will rarely exceed unity since this would require temperatures lower than − 160°C or higher than 80°C, neither of which are likely in the presence of ice under common conditions. Typically, $Ste \ll 1$ and therefore it is evident that corrections to $\beta(\tau, 0)$ will usually be small.

One rather curious implication of the first order term is the fact that whenever $\phi^\circ(\tau)$ passes through zero, the value of $\beta(\tau, Ste)$ at that time is given by $\beta(\tau, 0)$ whether *Ste* is zero or not. As noted elsewhere [1] a sinusoidal variation of surface temperature leaves the maximum depth of the ice–water interface unaltered by the Stefan number. Obviously, this result would apply to any periodic variation in surface temperature.

The requirement that the p-th component of the temperature (and its gradient) at the interface should be dependent upon the Stefan number was overlooked in [1]. Fortunately, this note reveals that the conclusions of [1] are unchanged and that the numerical error incurred in the temperature distribution given in [1] is very small.

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